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Remark on Some Functional Equations

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Abstract. In this note we give a new and simple proofs of some D. Adamović's results.

In papers [1], [2] and [3] D. Adamović consider functional equations:

$$f(f(x+y)) = f(x) + f(y) \tag{1}$$

and

$$f(x+y) = f(f(x)) + f(f(y)) \tag{2}$$

for any $x, y \in \mathbf{F}$, where $\mathbf{F} \in \{\mathbf{R}, \mathbf{C}\}$ and $f : \mathbf{F} \to \mathbf{F}$, which generalizes Cauchy equation. The aim of this note is to show that Adamović's result can be obtained by application of classical result of Pexider [5] (see also [4]).

Proposition 1. *(Pexider* [5]*) The general solution of functional equation*

$$f(x+y) = g(x) + h(y)$$

where $f, g, h : \mathbf{F} \to \mathbf{F}$, is given by

$$\begin{aligned} f(x) &= \varphi(x) + a + b \\ g(x) &= \varphi(x) + a \\ h(x) &= \varphi(x) + b, \end{aligned}$$

where $a, b \in \mathbf{F}$ are arbitrary constants and $\varphi : \mathbf{F} \to \mathbf{F}$ is arbitrary solution of Cauchy functional equation.

In fact, Propositions 1 and 2, in theirs forms given here, do not comprise the whole content of Adamović's result, which also gives the determination of idempotentical solution of the Cauchy equation (see remark). This momentum is not included in our consideration here.

Proposition 2 (Adamović [1]). *f is solution of equation* (1) *if and only if $f(x) = \varphi(x) + a$ where φ is idempotentical ($\varphi \circ \varphi = \varphi$) solution of Cauchy equation and $a \in \mathbf{F}$ is a constant such that $\varphi(a) = a$.*

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Proof. Let f be a solution of equation (1). Then, by application of Proposition 1 we have

$$f(x) = \varphi(x) + a \tag{3}$$

and

$$f(f(x)) = \varphi(x) + 2a, \tag{4}$$

where $a \in \mathbf{F}$ is a constant and $\varphi : \mathbf{F} \to \mathbf{F}$ is solution of Cauchy functional equation. By (3) and (4), we obtain

$$f(f(x)) = \varphi(\varphi(x) + a) + a = \varphi(\varphi(x)) + 2a, \tag{5}$$

and further

$$\varphi(\varphi(x)) + \varphi(a) + a = \varphi(x) + 2a. \tag{6}$$

For $x = 0$ we get

$$\varphi(a) = a, \tag{7}$$

such that (6) becomes

$$\varphi(\varphi(x)) = \varphi(x).$$

On the other hand, if f is given by (3) with arbitrary idempotentical solution φ of Cauchy functional equation and arbitrary constant $a \in \mathbf{F}$ satisfying (7), then by first equality in (5) and by the left side of (6),

$$\begin{aligned} f(f(x+y)) &= \varphi(\varphi(x+y)) + a) + a = (\varphi(x+y)) + 2a = \\ &= \varphi(x) + \varphi(y) + 2a = \varphi(x) + a + \varphi(y) + a = f(x) + f(y). \end{aligned}$$ □

Proposition 3 (Adamović [1]**.** *] f is solution of equation* (2) *if and only if it is idempotentical ($f \circ f = f$) solution of Cauchy equation.*

Proof. Let f be a solution of equation (2). Then we have, by Proposition 1

$$f(x) = \varphi(x) + 2a \tag{8}$$

and

$$f(f(x)) = \varphi(x) + a, \tag{9}$$

where $\varphi : \mathbf{F} \to \mathbf{F}$ is a solution of Cauchy functional equation and $a \in \mathbf{F}$ a constant. By (8) and (9),

$$\varphi(\varphi(x) + 2a) + 2a = \varphi(x) + a$$

and further

$$\varphi(\varphi(x)) + \varphi(a) + a = \varphi(x) + a. \tag{10}$$

For $x = 0$ we obtain

$$2\varphi(a) = -a, \tag{11}$$

such that (10) becomes

$$\varphi(\varphi(x)) = \varphi(x). \tag{12}$$

From (11) and (12) follows

$$\varphi(a) = \varphi(\varphi((a)) = \varphi(-\frac{a}{2}) = -\frac{1}{2}\varphi(a) = \frac{a}{4}.$$

Hence

$$-\frac{a}{2} = \frac{a}{4},$$

which implies $a = 0$. So we have $f = \varphi$ and consequently $f \circ f = f$.

On the other hand, a direct verification shows that every idempotentical solution of Cauchy equation satisfies equation (2). □

Remark 0.1. In Proposition 1 and 2 we gives the description of general solution of equations (1) and (2), by idempotentical solution of Cauchy equation, which complete determination was obtained by D. Adamović [1]. Some simple corrections of this result were given in [3].

Proposition 4 (Adamović [3])**.** *If $g : \mathbf{F} \to \mathbf{F}$ is idempotentical solution of Cauchy equation, then*

$$g(x) = \begin{cases} x, & \textit{for } x \in H_0 \\ \mathbf{L}(H_0), & \textit{for } x \in H \backslash H_0 \\ \sum_{h\in H} \alpha_h g(h), & \textit{for } x = \sum_{h \in H} \alpha_h h \end{cases}$$

where H is arbitrary Hamel base of $\mathbf{F}$, $\emptyset \neq H_0 \subseteq H$, $\mathbf{L}(H_0)$ is a linear hull of H, α_h rational numbers.

From this proposition follows that only continuous idempotentical solution of Cauchy equation is identical mapping.

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